

# The phase of a quantum mechanical particle in curved spacetime

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## Abstract

We investigate the quantum mechanical wave equations for free particles of spin 0, 1/2, 1 in the background of an arbitrary static gravitational field in order to explicitly determine if the phase of the wavefunction is  $S/\hbar = \int p_\mu dx^\mu/\hbar$ , as is often quoted in the literature. We work in isotropic coordinates where the wave equations have a simple manageable form and do not make a weak gravitational field approximation. We interpret these wave equations in terms of a quantum mechanical particle moving in medium with a spatially varying effective index of refraction. Due to the first order spatial derivative structure of the Dirac equation in curved spacetime, only the spin 1/2 particle has *exactly* the quantum mechanical phase as indicated above. The second order spatial derivative structure of the spin 0 and spin 1 wave equations yield the above phase only to lowest order in  $\hbar$ . We develop a WKB approximation for the solution of the spin 0 and spin 1 wave equations and explore amplitude and phase corrections beyond the lowest order in  $\hbar$ . For the spin 1/2 particle we calculate the phase appropriate for neutrino flavor oscillations.

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# 1 Introduction

The phase of a quantum mechanical particle in curved spacetime has been of considerable interest both theoretically and experimentally for many years. In the late 1970's researchers were interested in explaining the quantum mechanical interference fringes for neutrons traversing different paths in the Earth's gravitational field [1]. Recently, there has been renewed interest in this topic in order to investigate the interplay between gravitation and the quantum mechanical principle of linear superposition in relation to flavor oscillations of neutrinos in the context of the solar neutrino anomaly, and type-II supernova [2, 3, 4].

Most computations of the quantum mechanical phase (QMP) of a free particle in curved spacetime refer back to the seminal article by Stodolsky [5] who argued that the relativistically invariant phase  $S/\hbar$  would be given by

$$S(\mathbf{r}, t)/\hbar = \frac{1}{\hbar} \int_{\mathbf{r}_A, t_A}^{\mathbf{r}_B, t_B} p_\mu dx^\mu. \quad (1)$$

In Eq. (1)  $p_\mu = mg_{\mu\nu}u^\nu$  is the general relativistic 4-momentum, and  $u^\mu = dx^\mu/d\tau$  is the 4-velocity such that  $g_{\mu\nu}u^\mu u^\nu = 1$ . The phase is computed along a geodesic connecting the points  $(\mathbf{r}_A, t_A)$  and  $(\mathbf{r}_B, t_B)$ . The quantum mechanical particle is assumed to be a test particle in the sense that it moves in the background of the general relativistic metric and does not generate its own gravitational field. For any arbitrary metric in general relativity (GR), static or otherwise, we have a relationship amongst the momenta, called the *mass shell constraint*, given by

$$g^{\mu\nu} p_\mu p_\nu = (mc_0)^2, \quad (2)$$

where  $m$  is the rest mass of the particle and  $c_0$  is the vacuum speed of light. The form of the scalar quantum mechanical wave function proposed by Stodolsky was

$$\psi(\mathbf{r}, t) = A e^{iS(\mathbf{r}, t)/\hbar}, \quad (3)$$

where the amplitude  $A$  is assumed constant.

It is clear from Eq. (3) that a single spatial derivative of  $\psi(\mathbf{r}, t)$  can generate  $p_\mu$ , but that two spatial derivatives cannot generate the mass shell constraint Eq. (2), since imaginary cross terms involving gradients of  $p_\mu$  will be produced. Therefore, for second order wave equations, for which spin 0 and spin 1 particles are particular cases, the QMP cannot *exactly*

take the form of Eq. (1). In addition, the form of  $\psi(\mathbf{r}, t)$  in Eq. (3) must involve an amplitude change, which is well known from the standard WKB approximation of the wave function in non-relativistic quantum mechanics [6].

In this paper we examine the general relativistic wave equations for quantum mechanical particles of spin 0, 1/2, 1 in the background of a generic static metric of arbitrary strength which can be written in isotropic form. We do not make a weak gravitational field approximation, which is often the case in the literature [7]. By working in isotropic coordinates, the wave equations are expressed in simple forms, which are then easily interpreted. We explicitly demonstrate that for the Dirac equation in curved spacetime the form of the QMP is given exactly by Eq. (1), and that Eq. (2) is exactly satisfied. This result is directly traceable to the first order spatial derivative structure of the Dirac equation. The only difference from Stodolsky's proposal is that amplitude in Eq. (3) takes on a simple spatially varying form. For particles of spin 0 and spin 1, the wave equation is of second order in both time and space. We develop a WKB approximation for  $\psi(\mathbf{r}, t)$  which to lowest order in  $\hbar$  has the form of Eq. (1) and satisfies Eq. (2). We formally solve for  $\psi$  to all orders in  $\hbar$ . However, since the WKB expansion is only an asymptotic series [8] we only examine the next higher order phase and amplitude corrections. We give an interpretation of the above wave equations in terms of a particle moving in a medium with a spatially varying index of refraction  $n(\mathbf{r})$  as discussed in Evans *et. al.* [9] and Alsing [10] (see also [11]).

The outline of the paper is as follows. In Section 2 we briefly review Stodolsky's reasoning for the form of the phase of a quantum mechanical particle in curved spacetime as given in Eq. (1). In Section 3 we examine the general form of the quantum mechanical wave equation for a free particle in a background curved spacetime. In Section 4 we examine the Dirac equation in curved spacetime and show that in isotropic coordinates, it takes on a simple form even for arbitrary strength gravitational fields. In Section 5 we examine the wave equation for a scalar particle in curved spacetime and develop a WKB solution. We develop  $\hbar$ -dependent quantum corrections beyond the lowest order WKB amplitude and phase approximations found in standard textbooks. This WKB approximation has applicability to the curved spacetime spin 1 wave equation since the latter has the same form as the scalar wave equation for each of the vector components of the wave function. Throughout the discussion we draw the analogy to a particle moving in a medium with a spatially varying index of refraction, which the form of these wave equations explicitly exhibit. In Section 6 we relate the

classical optical-mechanical analogy of the lowest order approximation to the QMP. In Section 7 we calculate the QMP for a spin 1/2 particle appropriate for neutrino flavor oscillations in a gravitational field and relate it to the effective index of refraction discussed in the previous section. In Section 8 we summarize our results and discuss their implications in light of current calculations in the literature.

## 2 Stodolsky's proposal for the phase of a quantum mechanical particle in curved spacetime

In 1979, Stodolsky [5] argued the phase of a spinless quantum mechanical particle should take the form of Eq. (1). In the following we summarize his reasoning. In flat Minkowski spacetime elementary quantum mechanics tells us the phase of the particle's wavefunction is given by dimensionless quantity  $(\mathbf{p} \cdot \mathbf{x} - Et)/\hbar$ . The numerator of this expression has units of action,  $mc_0 \times$  (proper distance). Since the particle is taken to be a test particle, and therefore does not generate its own gravitational field, classically it would follow a geodesic. This is assumed to be true quantum mechanically as well, independent of the particle's spin. For an arbitrary metric  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ , Stodolsky proposed the phase  $\Phi$  accumulated by the particle in traversing the geodesic connecting the spacetime points  $A$  and  $B$  to be

$$\Phi = \frac{c_0}{\hbar} \int_A^B m ds \equiv S/\hbar. \quad (4)$$

The particle's quantum mechanical wavefunction is then taken to be proportional to the phase factor  $e^{i\Phi}$ . The integral appearing in Eq. (4) is just the relativistic action for a particle moving on a geodesic [12]. Stodolsky states ([5], p392), "If we wish the phase to be an invariant and to agree with elementary quantum mechanics this seems to be the only choice." Furthermore, dividing the metric by  $ds$  and defining  $p_\mu = mc_0 g_{\mu\nu} dx^\nu/ds$  as the canonical general relativistic momentum yields Eq. (1). Stodolsky finds additional reassurance from the observation that Eq. (1) also appears reasonable from the Feynman path integral approach to quantum mechanics. If Eq. (4) is valid for paths neighboring the classical path then one obtains in the usual way that the actual classical path is the one for which  $\delta\Phi = 0$ . But from the form of Eq. (4), this is just the classical relativistic condition for a geodesic. In his paper [5] Stodolsky examines Eq. (1) in the limit of weak gravitational fields for both static and stationary metrics.

The purpose of this work is to explicitly test whether Eq. (1) is a valid solution of the general relativistic wave equations for particles of spin 0, 1/2, 1. To this effect, we consider an arbitrary static gravitational field which can be written in isotropic coordinates [13]. For such gravitational fields, the spatial portion of the metric can be written in a conformally flat form,

$$ds^2 = \Omega^2(\mathbf{r}) c_0^2 dt^2 - \frac{1}{\Phi^2(\mathbf{r})} (dx^2 + dy^2 + dz^2) \equiv \Omega^2(\mathbf{r}) c_0^2 dt^2 - \frac{1}{\Phi^2(\mathbf{r})} |d\mathbf{r}|^2. \quad (5)$$

In the above,  $r$  is the isotropic radial marker.

Many static gravitational fields of physical interest can be written in the form of Eq. (5). The important class of static, spherically symmetric metrics written in the usual spherical coordinates  $\{t, r', \theta, \phi\}$  have the general, non-isotropic form

$$ds^2 = \frac{F(r')}{G(r')} c_0^2 dt^2 - \frac{dr'^2}{F(r')} - R^2(r') (d\theta^2 + \sin^2(\theta) d\phi^2). \quad (6)$$

For example, the Schwarzschild metric has  $F(r') = (1 - r'_s/r')$ ,  $G(r') = 1$ , and  $R(r') = r'$  where  $r'$  is the usual radial coordinate measured by an observer at infinity and  $r'_s = 2GM/c_0^2$  is the Schwarzschild radius. After transforming the Schwarzschild metric to isotropic coordinates  $\{t, r, \theta, \phi\}$  one obtains  $\Omega(r) = (1 + r_s/r)/(1 - r_s/r)$  and  $\Phi(r) = (1 + r_s/r)^{-2}$  where  $r_s = r'_s/2$ . The relationship between the coordinate radius  $r'$  and the isotropic radius  $r$  is given by  $r' = r \Phi^{-1}(r) = r (1 + r_s/r)^2$ . Other interesting static spherically symmetric metrics for both Einstein gravitation theory and non-Einstein gravity theories can be found in [14].

The utility of isotropic coordinates stems from the ease by which we can define an effective index of refraction for the gravitational field. By setting  $ds = 0$  in Eq. (5) we obtain the coordinate speed of light  $c(\mathbf{r})$  as

$$c(\mathbf{r}) = \left| \frac{d\mathbf{r}}{dt} \right| = c_0 \Omega(\mathbf{r}) \Phi(\mathbf{r}) \equiv \frac{c_0}{n(\mathbf{r})}, \quad (7a)$$

$$n(\mathbf{r}) \equiv \frac{1}{\Omega(\mathbf{r}) \Phi(\mathbf{r})}. \quad (7b)$$

The paths of both massive and massless particles in such metrics (i.e. geodesics) can be interpreted as motion through a medium of index with an effective refraction  $n(\mathbf{r})$ . For the Schwarzschild metric we have  $n(\mathbf{r}) = (1 + r_s/r)^3/(1 - r_s/r)$ . Evans *et al.* [9] used this concept to write the geodesic equations of motion for static metrics in Newtonian “F=ma” form. Alsing [10] later

extended this idea to the case of stationary metrics. In this paper, the quantity  $n(\mathbf{r})$  will continually arise in the wave equations developed, and will be naturally interpreted as an effective index of refraction.

For the metric given by Eq. (5) we can write the mass shell constraint Eq. (2) as

$$E^2 = \left( \frac{\mathbf{p}(\mathbf{r})c_0}{n(\mathbf{r})} \right)^2 + (\Omega(\mathbf{r}) m c_0^2)^2, \quad (8)$$

where we define the 4-momentum as  $p^\mu = (E/c_0, \mathbf{p})$ . This expression reduces to the ordinary flat space, special relativistic form  $E^2 = (\mathbf{p}c_0)^2 + (m c_0^2)^2$  as  $r \rightarrow \infty$ . This naturally leads us to propose the quantization rules

$$E = \hbar \omega, \quad \mathbf{p} = \hbar \mathbf{k}, \quad (9)$$

which define the frequency  $\omega$  and wave vector  $\mathbf{k}$ . Using Eq. (9) in Eq. (8), we can rewrite the mass shell constraint as

$$n^2(\mathbf{r}) k_0^2 + k^2(\mathbf{r}) = k_c^2(\mathbf{r}). \quad (10)$$

In Eq. (10) we have defined the 4-wave vector as  $k^\mu = (k_0, \mathbf{k})$  with  $k_0 = \omega/c_0$ ,  $k^2 = \mathbf{k} \cdot \mathbf{k}$  and  $k_c(\mathbf{r}) \equiv (\lambda_c \Phi(\mathbf{r}))^{-1}$  where  $\lambda_c = \hbar/(m c_0)$  is the usual Compton wavelength for a particle of mass  $m$ . Note the wave vector  $\mathbf{k}(\mathbf{r})$  and hence the velocity of the particle is position dependent as appropriate for a particle in a medium with index of refraction  $n(\mathbf{r})$ .

### 3 Quantum wave equations in curved spacetime

In practice, we only know how to quantize wave equations in flat Minkowski space [15]. Therefore, a natural way to describe curved spacetime is to erect local coordinate axes named *vierbeins* or *tetrads* [16, 17, 18] at each point  $X$  in spacetime and then project all tensor quantities onto these local, Lorentzian inertial frame axes. At each point  $X$  the local metric takes the flat spacetime form,  $\eta_{ab} = \text{diagonal } \{1, -1, -1, -1\}$ . Throughout this paper, Latin indices  $\{a, b, c, \dots\}$  near the beginning of the alphabet will refer to the local inertial frame with values  $\{0, 1, 2, 3\}$  while Greek indices  $\{\mu, \nu, \lambda, \dots\}$  will refer to the general coordinate system  $x^\mu$  with values  $\{0, 1, 2, 3\}$ . The tetrads  $e_\mu^a(x)$  and  $e_a^\mu(x)$  are defined by

$$g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab}, \quad g^{\mu\nu}(x) = e_a^\mu(x) e_b^\nu(x) \eta^{ab}. \quad (11)$$

Contravariant and covariant vectors  $V^\mu$  and  $V_\mu$  in the general coordinate system can be expressed as vectors  $V^a$  and  $V_a$  in the local inertial frame (and visa versa) by means of the transformations

$$\begin{aligned} V^\mu(x) &= V^a(x) e_a^\mu(x), & V^a(x) &= V^\mu(x) e_\mu^a(x), \\ V_\mu(x) &= e_\mu^a(x) V_a(x), & V_a(x) &= e_a^\mu(x) V_\mu(x). \end{aligned}$$

Local inertial frame indices are raised and lowered with the flat spacetime metric  $\eta_{ab}$ , while all general coordinate frame indices are raised and lowered with the metric  $g_{\mu\nu}$ .

The prescription for generalizing a flat spacetime wave equation to curved spacetime proceeds as follows: (i) begin with the appropriate flat spacetime Lagrangian for the wave equation of interest, (ii) replace all local partial coordinate derivatives by covariant derivatives via  $\partial_a \rightarrow e_a^\mu(x) \nabla_\mu$ , and (iii) contract all vectors, tensors, etc. into  $n$ -biens, ( $V^a(x) \rightarrow V^\mu(x) e_\mu^a(x)$ , etc.).

For a field  $\psi(x)$  of arbitrary spin the spin covariant derivative  $\nabla_\mu$  is defined by [16, 18, 19, 20]

$$\nabla_\nu \psi(x) = [\partial_\nu + \Omega_\nu(x)] \psi(x), \quad (12a)$$

$$\Omega_\nu(x) \equiv -\frac{i}{4} \omega_{ab\nu}(x) \sigma^{ab} = \frac{1}{8} \omega_{ab\nu}(x) [\gamma^a, \gamma^b], \text{ spin } 1/2 \quad (12b)$$

$$\Omega_\nu(x) \equiv \frac{1}{2} \omega_{ab\nu}(x) \Sigma^{ab}. \quad \text{arbitrary spin} \quad (12c)$$

In Eq. (12) the Fock-Ivanenko coefficients  $\Omega_\nu(x)$  (not to be confused with  $\Omega(\mathbf{r}) \equiv \sqrt{g_{00}(\mathbf{r})}$  in the isotropic metric Eq. (5)) are defined in terms of the spin connection coefficients  $\omega^a_{b\nu}$  given by

$$\omega^a_{b\nu} = e_\mu^a (e_b^\mu)_{;\nu} = e_\mu^a (\partial_\nu e_b^\mu + e_b^\sigma \Gamma_{\sigma\nu}^\mu), \quad (13a)$$

$$\omega_{ab\nu} = e^\mu_a e_{b\mu;\nu}, \quad \left( e^\mu_a \equiv g^{\mu\nu} \eta_{ab} e_\nu^b \right), \quad (13b)$$

which are antisymmetric in their Lorentzian indices  $\omega_{ba\nu} = -\omega_{ab\nu}$ . The semicolon denotes the usual Riemannian covariant derivative,  $V^\mu_{;\nu} \equiv \partial_\nu V^\mu + \Gamma^\mu_{\sigma\nu} V^\sigma$ . Since  $\eta_{ab;\sigma} = 0$  and the Riemannian metric compatibility condition gives  $g_{\mu\nu;\sigma} = 0$ , we can freely raise and lower both Lorentzian (Latin) and general coordinate indices (Greek) within a Riemannian covariant derivative operation. Note Eq. (13a) can be rearranged to define the action of the spin covariant derivative  $\nabla_\nu$  on the tetrad  $e_a^\mu$ ,

$$\nabla_\nu e_a^\mu \equiv \partial_\nu e_a^\mu + \Gamma^\mu_{\sigma\nu} e_a^\sigma - \omega^b_{a\nu} e_b^\mu = 0, \quad (14)$$

analogous to the Riemannian metric compatibility condition. Mnemonically, for the spin covariant derivative of a quantity  $T_{cd\dots\gamma\delta\dots}^{ab\dots\mu\nu\dots}$  with a mixed set of index types, each general coordinate index receives a contribution from the metric connection and each local Lorentzian index receives a contribution from the spin connection.

Finally, in the last expression Eq. (12c) the constant matrices  $\Sigma^{ab}$  are the generators of the Lorentz group for an arbitrary value of the spin [16, 18]. For the specific case of spin 1/2,  $\Sigma^{ab} = 1/4[\gamma^a, \gamma^b]$ , where  $\gamma^a$  are the constant Lorentzian gamma matrices, with an explicit representation given in Appendix A. In addition, we have used the conventional notation  $\sigma^{ab} = i/2[\gamma^a, \gamma^b]$  in Eq. (12b).

We are now ready to generalize the flat spacetime Lagrangian  $\mathcal{L}(x)$  for the fields of interest to curved spacetime. Since in flat spacetime the action is given by  $S_{flat} = \int d^4x \mathcal{L}(x)$ , the curved spacetime analogue of  $\mathcal{L}(x)$  must involve the volume factor  $\sqrt{-g(x)} = \det(e_\mu^a(x))$ , where  $g(x) \equiv \det(g_{\mu\nu}(x))$ , in order that  $S_{curved}$  transforms as a scalar.

### 3.1 Spin 0 field

For a spin 0 field  $\phi(x)$ ,  $\Sigma^{ab} = 0$ ,  $\partial_a \phi(x) \rightarrow e_a^\mu \nabla_\mu \phi(x)$  with  $\nabla_\mu \phi(x) = \partial_\mu \phi(x)$ , and we have

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{2} \left( \eta^{ab} \partial_a \phi \partial_b \phi - \frac{m^2 c_0^2}{\hbar^2} \phi \right) \\ &\rightarrow \frac{1}{2} (-g)^{1/2} \left( \eta^{ab} e_a^\mu \partial_\mu \phi e_b^\nu \partial_\nu \phi - \frac{m^2 c_0^2}{\hbar^2} \phi \right). \end{aligned} \quad (15)$$

Variation of the Lagrangian with respect to  $\phi(x)$  yields the general relativistic Klein-Gordon equation (GRKGE)

$$\begin{aligned} \left( \square + \frac{m^2 c_0^2}{\hbar^2} \right) \phi(x) &\equiv g^{\mu\nu} \phi_{;\mu;\nu} + (mc_0/\hbar)^2 \phi(x) \\ &= \left( g^{\mu\nu} \partial_\mu \partial_\nu - g^{\mu\nu} \Gamma_{\mu\nu}^\lambda \partial_\lambda + \frac{m^2 c_0^2}{\hbar^2} \right) \phi(x) = 0, \end{aligned} \quad (16)$$

where we have defined the covariant Laplace-Beltrami operator  $\square \phi(x) \equiv g^{\mu\nu} \phi_{;\mu;\nu}$ . For the case of spin 0 we can actually derive Eq. (16) much more directly from Eq. (2). Classically, the Lagrangian for geodesic motion is given by  $L = m c_0 ds = m c_0 \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$ , where  $\dot{x}^\mu = dx^\mu/d\tau$ ,  $d\tau = ds/c_0$ . The Hamiltonian is given by  $\sum_\mu (\partial L / \partial \dot{x}^\mu) \dot{x}^\mu - L \equiv 0$ . This implies there



must be a constraint amongst the momenta, which is given by Eq. (2), and acts as the effective Hamiltonian for the system [21],  $H_{eff} = g^{\mu\nu} p_\mu p_\nu$ . Substituting in the quantization condition  $p_\mu \rightarrow -i\hbar\nabla_\mu$  yields Eq. (16) directly.

Note the most general Lagrangian allows for a term proportional to the Ricci scalar  $R(x) = R_\mu{}^\mu(x)$ , i.e. an extra term  $-\xi R(x) \phi^2(x)$  in Eq. (15), which leads to a corresponding term  $\xi R(x) \phi(x)$  in the GRKGE [16]. The value of  $\xi$  is not determined by any physical principle and to date has not been experimentally measured due to the minute effects of curvature in our solar system. The value of  $\xi = 0$  is called *minimal coupling* for obvious reasons. In  $n$  dimensions the value of  $\xi(n) = 1/4(n-2)/(n-1)$ , which assumes the value of  $\xi = 1/6$  in four dimensions, is called *conformal coupling*. For massless particles under conformal coupling the GRKGE is invariant under the simultaneous conformal transformation of the metric  $g_{\mu\nu}(x) \rightarrow f^2(x) g_{\mu\nu}(x)$  and field transformation  $\phi(x) \rightarrow f^{-1}(x) \phi(x)$  for an arbitrary function  $f(x)$ . In this paper we concern ourselves with regions of spacetime devoid of matter so that Einstein's equations reduce to  $R_{\mu\nu} = 0$  and consequently  $R(x) = 0$ .

### 3.2 Spin 1/2 field

For the case of spin 1/2 we have  $\Sigma^{ab} = 1/4[\gamma^a, \gamma^b] = -i/2\sigma^{ab}$ ,  $\partial_a\psi \rightarrow e_a{}^\mu\nabla_\mu\psi$ , and we obtain

$$\begin{aligned} \mathcal{L}(x) &= \frac{i}{2} \left( \bar{\psi} \gamma^a \partial_a \psi - (\partial_a \bar{\psi}) \gamma^a \psi - \frac{mc_0}{\hbar} \bar{\psi} \psi \right) \\ &\rightarrow (-g)^{1/2} \frac{i}{2} \left( \bar{\psi} \gamma^a e_a{}^\mu \nabla_\mu \psi - e_a{}^\mu (\nabla_\mu \bar{\psi}) \gamma^a \psi - \frac{mc_0}{\hbar} \bar{\psi} \psi \right) \quad (17a) \end{aligned}$$

$$\equiv (-g)^{1/2} \frac{i}{2} \left( \bar{\psi} \gamma^\mu \nabla_\mu \psi - (\nabla_\mu \bar{\psi}) \gamma^\mu \psi - \frac{mc_0}{\hbar} \bar{\psi} \psi \right) \quad (17b)$$

In the above we have defined the curved spacetime counterparts to the Dirac matrices as  $\gamma^\mu(x) = \gamma^a e_a{}^\mu(x)$  which satisfy the Clifford algebra  $\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x)$ . Variation of Eq. (17b) with respect to  $\bar{\psi}(x) = \psi^\dagger(x) \gamma^0(x)$  yields the general relativistic Dirac equation (GRDE) for spin 1/2 particles

$$\left( i \gamma^\mu(x) \nabla_\mu - \frac{mc_0}{\hbar} \right) \psi(x) = 0. \quad (18)$$

Note that no geometric curvature term with a dimensionless coefficient can be added to the Lagrangian and thus to the GRDE [16].

### 3.3 Comment on the Pauli-Schrödinger equation

In flat spacetime, the Dirac wavefunction also satisfies the Klein-Gordon equation due to the factorization  $(i\gamma^a\partial_a + mc_0/\hbar)(i\gamma^a\partial_a - mc_0/\hbar)\psi = 0$  which implies  $[\eta^{ab}\partial_a\partial_b + (mc_0/\hbar)^2]\psi = 0$ . However, in the presence of a gravitational field, this is no longer the case [22]. Using Eq. (18), the analogous calculation produces  $(i\gamma^\mu\nabla_\mu + mc_0/\hbar)(i\gamma^\nu\nabla_\nu - mc_0/\hbar)\psi = 0 \Rightarrow [\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu + (mc_0/\hbar)^2]\psi = 0$ . This can be written as  $[g^{\mu\nu}\nabla_\mu\nabla_\nu - 1/2\sigma^{\mu\nu}K_{\mu\nu} + (mc_0/\hbar)^2]\psi = 0$ , where  $K_{\mu\nu} \equiv 1/2(\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu) = \partial_\nu\Omega_\mu - \partial_\mu\Omega_\nu + [\Omega_\nu, \Omega_\mu]$  is called the *spin curvature* in analogy to the Riemann curvature [23]. The relation between the spin curvature and the Riemann curvature is given by (see appendix of [22])  $K_{\mu\nu} = 1/4R_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}$  and  $R\mathbf{I} = R_{\mu\nu\alpha\beta}\sigma^{\mu\nu}\sigma^{\alpha\beta} = -2K_{\mu\nu}\sigma^{\mu\nu}$  where  $R$  is the Ricci scalar and  $\mathbf{I}$  is the  $4 \times 4$  unit matrix. Using the trace of Einstein's equations  $R_{\mu\nu} - 1/2g_{\mu\nu}R = -8\pi G/c_0^2 T_{\mu\nu}$  we obtain

$$\left[ g^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{2\pi G}{c_0^2} T + (mc_0/\hbar)^2 \right] \psi = 0, \quad (19)$$

where  $T = T_\mu{}^\mu$  is the trace of the energy-momentum tensor.

Eq. (19) is the generally covariant extension of the Pauli-Schrödinger equation [24] and describes spin 1/2 particles *in a gravitational field*. As such, the covariant derivative in Eq. (19) is given by Eq. (12a) involving the Fock-Ivanenko coefficients. On the other hand, Eq. (16) is the generally covariant extension of the Klein-Gordon equation for spin 0 particles, with no  $\Omega_\mu$  terms. If we expand the spin covariant derivatives in Eq. (19) using Eq. (12a) we have

$$\begin{aligned} & [g^{\mu\nu}\partial_\mu\partial_\nu + (mc_0/\hbar)^2] \psi + (2\pi G/c_0^2)^2 T \psi \\ & + g^{\mu\nu}(2\Omega_\mu\partial_\nu + (\partial_\mu\Omega_\nu) + \Omega_\mu\Omega_\nu)\psi = 0. \end{aligned} \quad (20)$$

In flat spacetime and in regions where  $T = 0$ , the tetrads are constant unit vectors and  $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$  in Cartesian coordinates. Hence, the Fock-Ivanenko coefficients are zero. The remaining terms in the square brackets above constitute the usual Minkowski Klein Gordon equation for the spinor field  $\psi(x)$ . Similarly, for the scalar wave equation Eq. (16) in flat spacetime in Cartesian coordinates, the affine connection  $\Gamma^\sigma_{\mu\nu}$  is also zero, and so again we recover the Minkowski Klein Gordon equation, for the scalar field  $\phi(x)$ . The point to be made here is that in flat spacetime the Dirac wave function (which describes spin 1/2 particles) is also a solution of the Klein-Gordon equation (which describes spin 0 particles) by the virtue of there being *no*

*gravitational field* [22]. In curved spacetime, this property no longer holds. In the presence of a gravitational field, the solution to the generally covariant Dirac equation is a solution of the generally covariant Pauli-Schrödinger equation.

### 3.4 Massless spin 1 field

In Minkowski spacetime the electromagnetic (massless, spin 1) field  $F^{ab}$  is described by the Lagrangian  $\mathcal{L} = -1/4 F^{ab} F_{ab}$ . where  $F_{ab} = \partial_b A_a - \partial_a A_b$  in terms of the vector potential  $A_a$ . The gauge freedom of the vector potential prevents a straightforward quantization of the theory and leads to the introduction of gauge-fixing terms  $\mathcal{L}_G = -1/2 \zeta^{-1} (\partial_a A^a)^2$  where  $\zeta$  is a parameter determining the choice of the gauge. In the *Feynman gauge*  $\zeta = 1$ , variation of the combined action  $\mathcal{L} + \mathcal{L}_G$  with respect to the vector potential leads to the wave equations  $\square A_c \equiv \eta_{ab} \partial_a \partial_b A_c = 0$  for each component of the vector potential.

In curved spacetime the electromagnetic field takes on the same form as in Minkowski spacetime due to the cancellation of the connection terms,  $F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ . To generalize the Maxwell field to curved spacetime we make the substitution  $A_a \rightarrow A_\mu$ ,  $\partial_a \rightarrow e_a^\mu \nabla_\mu$  with  $\Omega_\mu(x)$  given by Eq. (12c) with  $[\Sigma_{ab}]_c^d = \delta_a^d \eta_{bc} - \delta_b^d \eta_{ac}$  corresponding to the  $(1/2, 1/2)$  representation of the Lorentz group. The action takes the form of  $\mathcal{L} = -1/4 F^{\mu\nu} F_{\mu\nu}$  and correspondingly the gauge fixing term takes the form  $\mathcal{L}_G = -1/2 \zeta^{-1} (\nabla_\mu A^\mu)^2$ . Variation of the action  $\mathcal{L} + \mathcal{L}_G$  leads to the wave equations

$$\begin{aligned} A_{\mu;\nu}{}^\nu + R_\mu{}^\nu A_\nu - (1 - \zeta^{-1}) A_{\nu;\mu}{}^\nu &= 0, \\ \square A_\mu &= 0, \quad \text{Feynman gauge and } R_{\mu\nu} = 0 \end{aligned} \quad (21)$$

For our purposes, each component of the vector potential  $A_\mu(x)$  in Eq. (21) satisfies the massless GRKGE, so that it suffices for us to examine the spin 0 GRKGE Eq. (16).

## 4 Plane Wave-like solutions for the GRDE

We are now interested in finding solutions to the GRDE of the form Eq. (3) which we write as  $\psi(\mathbf{r}, t) = A(\mathbf{r}, t) e^{iS(\mathbf{r}, t)/\hbar}$  where  $u$  is a Dirac spinor. Writing out Eq. (18) and using Eq. (12b) the GRDE is explicitly given as

$$\left[ i\gamma^c e_c{}^\mu \left( \partial_\mu - \frac{i}{4} \sigma^{ab} e_a{}^\nu e_{b\nu;\mu} \right) - \frac{mc_0}{\hbar} \right] \psi(x) = 0. \quad (22)$$

Using the explicit representation of the Minkowski spacetime Dirac matrices  $\gamma^a$  given in the Appendix A, substituting in the static metric Eq. (5) and dividing the resulting equation by  $\Phi(\mathbf{r})$  yields after some algebra [25]

$$\left[ i \left( n(\mathbf{r}) \gamma^0 \partial_0 + \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + \frac{3}{4} \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \zeta(x) \right) - k_c(\mathbf{r}) \right] \psi(x) = 0. \quad (23)$$

In Eq. (23) we defined  $\boldsymbol{\nabla} = \{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$  as the spatial gradient in the general coordinate frame and  $\boldsymbol{\gamma} = \{\gamma^1, \gamma^2, \gamma^3\}$  the spatial components of the constant Dirac matrices in the local Lorentz frame. The quantity  $\zeta(x)$  is given by  $\zeta(x) \equiv \ln[\Omega(\mathbf{r})/\Phi^2(\mathbf{r})]$ .

The middle term in Eq. (23)  $\frac{3}{4} \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \zeta(x)$  can be removed by defining the wavefunction as  $\psi(\mathbf{r}, t) = f(\mathbf{r}) \phi(\mathbf{r}, t)$ . A simple calculation reveals  $f(\mathbf{r}) = e^{-3/4 \zeta(x)} = (\Phi^2(\mathbf{r})/\Omega(\mathbf{r}))^{3/4}$ . The resulting equation for  $\phi(x)$  is given by

$$\left[ i \left( \gamma^0 \frac{n(\mathbf{r})}{c_0} \frac{\partial}{\partial t} + \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \right) - k_c(\mathbf{r}) \right] \phi(\mathbf{r}, t) = 0. \quad (24)$$

Eq. (24) is directly interpretable as the Dirac equation in flat spacetime with a spatially varying index of refraction  $n(\mathbf{r})$  and a spatially varying Compton wavelength  $\lambda_c(\mathbf{r}) = \lambda_c^{flat} \Phi(\mathbf{r})$ ,  $k_c(\mathbf{r}) = 1/\lambda_c(\mathbf{r})$ ,  $\lambda_c^{flat} = \hbar/mc_0$ .

We now seek a plane wave-like solution by substituting in Stodolsky's suggestion for the QMP in the form

$$\phi(\mathbf{r}, t) = A \exp \left[ \frac{i}{\hbar} \left( \int^{\mathbf{r}} \mathbf{p}(\mathbf{x}) \cdot d\mathbf{x} - E t \right) \right] \quad (25)$$

where  $A$  is a constant Dirac spinor. Note that  $E$  is a constant of the geodesic motion given by  $E \equiv p_0 c_0 = \dot{t} \Omega^2(\mathbf{r}) m c_0^2$  since the metric is independent of the coordinate  $t$ . Separating Eq. (24) into a coupled set of two-spinor equations we have

$$\begin{pmatrix} n(\mathbf{r}) E/c_0 - m c_0/\Phi(\mathbf{r}) & -\boldsymbol{\sigma} \cdot \mathbf{p}(\mathbf{r}) \\ -\boldsymbol{\sigma} \cdot \mathbf{p}(\mathbf{r}) & n(\mathbf{r}) E/c_0 + m c_0/\Phi(\mathbf{r}) \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} = 0, \quad (26)$$

where  $\boldsymbol{\sigma}$  are the usual  $2 \times 2$  Pauli spin matrices (Appendix A). Eq. (26) is a homogeneous set of algebraic equations and therefore only has a nontrivial solution if the determinant of the coefficients is zero. Using the identity  $(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mathbf{p} \cdot \mathbf{p} \equiv \mathbf{p}^2$  and setting the determinant of the coefficients equal to zero yields precisely the mass shell constraint in the form of Eq. (8).

To summarize, we have found an exact plane wave-like solution for the GRDE for static metrics in isotropic coordinates to be of the form

$$\begin{aligned}\psi(\mathbf{r}, t) &= A \left( \frac{\Phi^2(\mathbf{r})}{\Omega(\mathbf{r})} \right)^{3/4} \exp \left[ \frac{i}{\hbar} \left( \int^{\mathbf{r}} \mathbf{p}(\mathbf{x}) \cdot d\mathbf{x} - E t \right) \right] \\ &= A \left( \frac{\Phi^2(\mathbf{r})}{\Omega(\mathbf{r})} \right)^{3/4} \exp \left( i \int^{\mathbf{r}} k_\mu dx^\mu \right).\end{aligned}\quad (27)$$

This solution is normalizable with respect to the inner product

$$\int d^3\mathbf{r} \sqrt{\det g_{ij}} \psi^\dagger(\mathbf{r}, t) \psi(\mathbf{r}, t), \quad (28)$$

where  $\det g_{ij} = \Phi^{-6}(\mathbf{r})$  is the determinant of the spatial portion of the metric  $g_{\mu\nu}$ .

Eq. (27) is one of the main results of this paper. It is an exact solution, valid for arbitrary strength gravitational fields for both massive and massless fields. Several authors [7, 26, 27, 28, 29] begin with Eq. (22) but examine the equation in the weak field limit  $e_a{}^\mu = \delta_a{}^\mu + 1/2 h_a{}^\mu$ . In some of those works, a Foldy-Wouthuysen transformation [30, 31] is performed to write the GRDE in the form of an effective weak field Schrödinger equation,  $i\hbar \partial\psi/\partial t = H\psi$ . Typically, one is interested in the Hamiltonian with respect to the measure  $\int d^3\mathbf{r}$  (versus the measure in Eq. (28)) so that the momentum operator can be interpreted in the usual flat spacetime form as a spatial gradient in Cartesian coordinates,  $\mathbf{p} = -i\hbar \nabla$ . In such cases, a new wave function is defined by  $\tilde{\psi} = (\det g_{ij})^{1/4} \psi \equiv U\psi$ , with the corresponding Hamiltonian  $\tilde{H} = UH U^{-1}$ . The focus of the above works is for the most part, the physical interpretation of terms in the effective weak field Hamiltonian  $\tilde{H}$  as post-Newtonian corrections to the gravitational potential, with an emphasis on spin-gravity coupling.

Instead, in this work, we are interested in the solution of the GRDE Eq. (22) directly, with emphasis on strong gravitational fields and the issue of whether or not the phase of the wavefunction takes the form proposed by Stodolsky. If one were to try force an interpretation of Eq. (24) as a Schrödinger equation, the effective Hamiltonian would contain non-Hermetian terms. This fact is well known (see [26, 27]) and arises because of our removal of terms that involved gradients of the gravitational field, and the introduction of a spatially varying function  $n(\mathbf{r})$  in front of the first order time derivative. As stated earlier, we interpret Eq. (24) as the flat spacetime Dirac equation for a particle moving in a medium with a spatially varying effective index of refraction and Compton wavelength. Regardless

of any imposed physical interpretation, the point we wish to emphasize here is that the GRDE admits a solution with exactly the phase suggested by Stodolsky Eq. (1), and a spatially varying amplitude.

## 5 WKB Solution to the GRKGE

To develop a solution for the GRKGE Eq. (16), we begin by using the identity [12]  $g^{\mu\nu}\Gamma_{\mu\nu}^\lambda = -(-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\lambda}]$ . Substituting this and the static metric Eq. (5) into Eq. (16) yields

$$\frac{n^2}{c_0^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi - \nabla\xi \cdot \nabla\psi + k_c^2(\mathbf{r})\psi = 0, \quad (29)$$

where we define the quantity  $\xi \equiv \ln(\Omega(\mathbf{r})/\Phi(\mathbf{r}))$ . As in the previous section, we can remove the term linear in  $\nabla\psi$  by seeking a solution of the form

$$\psi(\mathbf{r}, t) = f(\mathbf{r})\phi(\mathbf{r}, t), \quad (30)$$

with  $f(\mathbf{r}) = (\Phi(\mathbf{r})/\Omega(\mathbf{r}))^{1/2}$  and  $\phi(\mathbf{r}, t)$  satisfying

$$\frac{n^2}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + [k_c^2(\mathbf{r}) + \eta(\mathbf{r})]\phi = 0, \quad (31)$$

where

$$\eta(\mathbf{r}) \equiv \frac{1}{2}\nabla^2\xi(\mathbf{r}) + \frac{1}{4}|\nabla\xi(\mathbf{r})|^2. \quad (32)$$

Note Eq. (31) is essentially the flat spacetime wave equation for a scalar particle moving in spatially varying index of refraction and Compton wavelength. However, there is an additional term  $\eta(\mathbf{r})$  in the wave equation which is a direct result of the geometric cross term  $g^{\mu\nu}\Gamma_{\mu\nu}^\lambda$  arising from the product of the covariant derivatives. In the weak gravitation field limit this term is typically dropped [7] when seeking an approximate solution to the wave equation. This term can also be exactly eliminated by transforming to *harmonic* coordinates ([18] p161-163) for which the identity above becomes four coordinate conditions, namely  $\partial_\mu[(-g)^{1/2}g^{\mu\lambda}] = 0$ . However, the use of harmonic coordinates introduces off-diagonal terms in the metric, which for the Schwarzschild case ([18], p181) take the form of  $h(R)\mathbf{X}\cdot d\mathbf{X}$  where  $\mathbf{X}$  are the new harmonic Cartesian coordinates. In this work, we prefer to remain in isotropic coordinates, where the metric has a simple diagonal form and its spatial portion is Euclidean conformally flat, and remove the cross term

$-\nabla\xi \cdot \nabla\psi$  arising from  $g^{\mu\nu}\Gamma_{\mu\nu}^\lambda$  by the transformation of the wavefunction, Eq. (30).

A direct substitution of a plane wave-like solution of the form of Eq. (25) *does not* yield the mass shell constraint Eq. (8). This is due in part to the existence of the term  $\eta(\mathbf{r})$  and the fact that the spatial Laplacian  $\nabla^2$  generates imaginary terms linear in  $\nabla \cdot \mathbf{p}$ . In the next section we develop a WKB solution to Eq. (31).

### 5.1 The WKB expansion

Let us look for a stationary solution to Eq. (31) of the form

$$\phi(\mathbf{r}, t) = A \exp \left[ \frac{i}{\hbar} \mathcal{S}(\mathbf{r}, t) \right] \quad (33)$$

with  $A$  constant and where we take the phase to be of the form

$$\mathcal{S}(\mathbf{r}, t) = S(\mathbf{r}) - Et. \quad (34)$$

Substitution of Eq. (33) and Eq. (34) into Eq. (31) yields

$$\left( p^2 - (\nabla S)^2 \right) + i\hbar \nabla^2 S - \hbar^2 \eta(\mathbf{r}) = 0, \quad (35)$$

where we have used Eq. (8).

We now expand  $S(\mathbf{r})$  in a power series expansion in  $\hbar$  via

$$S(\mathbf{r}) = \sum_{n=0}^{\infty} \hbar^n S_n(\mathbf{r}). \quad (36)$$

Substituting this into Eq. (35) yields the set of equations

$$O(\hbar^0) : \quad (\nabla S_0)^2 - p^2 = 0 \quad (37a)$$

$$O(\hbar^1) : \quad 2\nabla S_0 \cdot \nabla S_1 = i\nabla^2 S_0 \quad (37b)$$

$$O(\hbar^2) : \quad 2\nabla S_0 \cdot \nabla S_2 = i\nabla^2 S_1 - (\nabla S_1)^2 + \eta(\mathbf{r}) \quad (37c)$$

$$O(\hbar^{n \geq 3}) : \quad 2\nabla S_0 \cdot \nabla S_n = i\nabla^2 S_{n-1} - \sum_{j=1}^{n-1} \nabla S_j \cdot \nabla S_{n-j} \quad (37d)$$

## 5.2 The eikonal equation

Eq. (37a) is the dominant contribution to the phase of the wavefunction and represents the eikonal equation. We can solve this equation for  $S_0$  via

$$S_0(\mathbf{r}) = \int^{\mathbf{r}} \mathbf{p}(\mathbf{x}) \cdot d\mathbf{x} \quad (38)$$

where

$$p = |\mathbf{p}| \equiv \frac{E}{c_0} N(\mathbf{r}) = \frac{E}{c_0} n(\mathbf{r}) \sqrt{1 - \left( \frac{\Omega(\mathbf{r}) m c_0^2}{E} \right)^2} = \frac{E}{c_0} \frac{n^2(\mathbf{r}) v(\mathbf{r})}{c_0}, \quad (39)$$

$$\mathbf{p}(\mathbf{r}) = \frac{E}{c_0} \frac{n^2 \mathbf{v}(\mathbf{r})}{c_0}. \quad (40)$$

In Eq. (39) we have introduced several definitions. First, the magnitude  $p$  of the spatial momentum is obtained by rearranging Eq. (8). Since the energy  $E$  is a constant of the motion for a static metric, the coordinate velocity  $\mathbf{v}(\mathbf{r}) = d\mathbf{r}/dt$  is a function of position. Its magnitude is given by [9]  $v(\mathbf{r}) = c_0/n(\mathbf{r}) \sqrt{1 - (\Omega(\mathbf{r}) m c_0^2/E)^2}$  and is related to  $p$  as given above. Note that for light  $m = 0$  and  $v(\mathbf{r}) = c_0/n(\mathbf{r})$  which allows the interpretation of  $n(\mathbf{r})$  as the index of refraction for massless particles. For massive particles  $E$  is an extra degree of freedom which specifies the initial speed of the particle. The quantity

$$N(\mathbf{r}) = n(\mathbf{r}) \sqrt{1 - \left( \frac{\Omega(\mathbf{r}) m c_0^2}{E} \right)^2} = \frac{n^2(\mathbf{r}) v(\mathbf{r})}{c_0}, \quad (41)$$

may be interpreted as the index of refraction for massive de Broglie waves. We will return to this point in Section 6.

$S_0(\mathbf{r}) = \text{constant}$  are just the wave front surfaces with normal given by

$$\nabla S_0(\mathbf{r}) = \mathbf{p}(\mathbf{r}) \quad \text{or} \quad \frac{\nabla S_0(\mathbf{r})}{\hbar} = \frac{\mathbf{p}(\mathbf{r})}{\hbar} \equiv \frac{1}{\lambda(\mathbf{r})}, \quad (42)$$

where  $\lambda(\mathbf{r})$  is the de Broglie wavelength of the particle. Following Holmes [32], let us characterize the wavefronts  $S_0(\mathbf{r})$  by coordinates  $\mathbf{r} = \bar{\mathbf{x}}(l, \alpha, \beta)$  where  $l$  is arc length along the ray trajectory normal to surfaces of constant  $S_0$ , and  $\alpha$  and  $\beta$  are coordinates used to parameterize the wavefront surfaces (for e.g. spherical coordinates). We write the unit tangent vector to the ray  $d\bar{\mathbf{x}}/dl$  as a vector in the direction of  $\nabla S_0(\mathbf{r})$  via

$$\frac{d\bar{\mathbf{x}}}{dl} = \frac{\nabla S_0(\mathbf{r})}{|\nabla S_0(\mathbf{r})|} = \frac{\mathbf{p}(\mathbf{r})}{p}. \quad (43)$$



Note we can write

$$\frac{dS_0}{dl} = \frac{d\bar{\mathbf{x}}}{dl} \cdot \nabla S_0 = |\mathbf{p}(\mathbf{r})| \equiv p. \quad (44)$$

Thus, another way to write Eq. (38) is

$$S_0(l, \alpha, \beta) = \int^l p \, dl. \quad (45)$$

### 5.3 The transport equation

We now need to solve the transport equation Eq. (37b),  $2\nabla S_0 \cdot \nabla S_1 = i\nabla^2 S_0$ . We note for any function  $F$ , we can write

$$dF/dl = d\bar{\mathbf{x}}/dl \cdot \nabla F = 1/p \nabla S_0 \cdot \nabla F \quad (46)$$

from the ray equation Eq. (43). Thus, substituting  $\nabla S_0 \cdot \nabla S_1 = p dS_1/dl$  into the transport equation gives us a first order differential equation for  $S_1(l, \alpha, \beta)$ ,

$$\frac{dS_1}{dl} = \frac{i}{2p} \nabla^2 S_0,$$

with solution

$$S_1(l, \alpha, \beta) = \frac{i}{2} \int^l dl \frac{\nabla^2 S_0}{p}. \quad (47)$$

A short calculation shown in Appendix B, reveals

$$\frac{\nabla^2 S_0}{p} = \frac{d(pJ)/dl}{pJ}, \quad (48)$$

where

$$J = \left| \frac{\partial \mathbf{x}}{\partial(l, \alpha, \beta)} \right|, \quad (49)$$

is the Jacobian of the transformation from the curvilinear ray coordinates  $(l, \alpha, \beta)$  to Cartesian coordinates. To prove Eq. (49) one needs to show (see Appendix B)

$$\partial_l J = J \nabla \cdot \left( \frac{\nabla S_0}{p} \right) = J \nabla \cdot \frac{d\bar{\mathbf{x}}}{dl}, \quad (50)$$

where  $d\mathbf{x}/dl$  is the unit tangent to the particle's trajectory, normal to surfaces of constant  $S_0$ . The net result is that upon substitution of Eq. (48) into Eq. (47), one can perform the integral to obtain

$$S_1(\mathbf{r}(l, \alpha, \beta)) = \frac{i}{2} \ln \left( \frac{p(\mathbf{r})J(\mathbf{r})}{p(\mathbf{r}_0)J(\mathbf{r}_0)} \right) \equiv \frac{i}{2} \mu(\mathbf{r}) \quad (51)$$

where we define  $\mu(\mathbf{r}) \equiv \ln \left( \frac{p(\mathbf{r})J(\mathbf{r})}{p(\mathbf{r}_0)J(\mathbf{r}_0)} \right)$ , and  $\mathbf{r}_0 = \mathbf{r}|_{l=0}$ . To the lowest order correction  $O(\hbar^0)$  in the phase and amplitude, we have found the WKB approximate solution

$$\begin{aligned} \psi_1(\mathbf{r}, t) &= \sqrt{\frac{\Phi(\mathbf{r})}{\Omega(\mathbf{r})}} \phi_1(\mathbf{r}, t) \\ &= A \sqrt{\frac{\Phi(\mathbf{r})}{\Omega(\mathbf{r})}} \sqrt{\frac{p(\mathbf{r}_0)J(\mathbf{r}_0)}{p(\mathbf{r})J(\mathbf{r})}} \exp \left[ \frac{i}{\hbar} \left( \int^{\mathbf{r}} \mathbf{p}(\mathbf{x}) \cdot d\mathbf{x} - Et \right) \right] \end{aligned} \quad (52)$$

where the subscript 1 on  $\psi_1(\mathbf{r}, t)$  indicates that we have carried out the WKB expansion to  $S_{n=1}$ . Note that Eq. (52) does take into account, to lowest order, the term  $\xi(\mathbf{r}) \equiv \ln(\Omega(\mathbf{r})\Phi^{-1}(\mathbf{r}))$  which arises from the covariant derivative term  $g^{\mu\nu}\Gamma_{\mu\nu}^\lambda$ . What has been left out are higher order terms involving  $\nabla^2 \xi(\mathbf{r})$ , and  $\nabla \xi(\mathbf{r})$  in  $\eta(\mathbf{r})$ , which come from the quantum corrections terms,  $S_{n \geq 2}$  (which we deal with next). Note that, already this solution is valid for strong gravitational fields (as opposed to only weak fields as considered by Donoghue and Holstein [7]) with the restriction that  $p^2 \gg \hbar^2 \eta(\mathbf{r})$ , corresponding to

$$\lambda(\mathbf{r}) \left\{ |\nabla \xi(\mathbf{r})|, \left| \sqrt{\nabla^2 \xi(\mathbf{r})} \right| \right\} \ll 1. \quad (53)$$

Recall that  $\xi(\mathbf{r}) \equiv \ln(\Omega(\mathbf{r})\Phi^{-1}(\mathbf{r}))$  so that the above condition can still possibly hold for reasonable distances close to horizon of a black hole, say. We investigate this in the next section.

#### 5.4 Estimation of terms in Schwarzschild metric

If we had defined  $\phi(\mathbf{r}, t) = u(\mathbf{r}) \exp(-iEt/\hbar)$  in our wave equation Eq. (31) the result would have been an equation of the Helmholtz form

$$\nabla^2 u(\mathbf{r}) + k^2(\mathbf{r}) \left( 1 - \frac{\eta(\mathbf{r})}{k^2(\mathbf{r})} \right) u(\mathbf{r}) = 0 \quad (54)$$

where  $p(\mathbf{r}) = \hbar k(\mathbf{r})$ ,  $k(\mathbf{r}) = k_0 N$ , with  $k_0 = \omega_0/c_0 = E/\hbar c_0$ . Therefore we want to consider the order of magnitude of the terms  $|\nabla \xi(\mathbf{r})|^2/k^2(\mathbf{r})$  and  $|\nabla^2 \xi(\mathbf{r})|^2/k^2(\mathbf{r})$  with  $\xi(\mathbf{r}) \equiv \ln(\Omega(\mathbf{r})\Phi^{-1}(\mathbf{r}))$ .

For the Schwarzschild metric in isotropic coordinates we have

$$\begin{aligned}\Omega &= \frac{1 - 1/\rho}{1 + 1/\rho}, \quad \Phi = \frac{1}{(1 + 1/\rho)^2}, \quad \xi = \ln(1 - \frac{1}{\rho^2}), \\ n &= \frac{(1 + 1/\rho)^3}{(1 - 1/\rho)}, \quad k = k_0 n \sqrt{1 - \frac{\Omega^2}{E'^2}}\end{aligned}\tag{55}$$

where  $E'^2 \equiv E/(mc_0^2)$  and  $\rho = r/r_s$ , with  $r_s = GM/c_0^2 = 0.74 M/M_\odot$  km (the the gravitational radius).

For massless particles,  $E' \rightarrow \infty$ , so that  $k = k_0 n$  and we obtain

$$\text{massless : } \frac{|\nabla \xi(\mathbf{r})|^2}{k^2(\mathbf{r})} = \frac{4}{(k_0 r_s)^2} \frac{1}{\rho^6} \frac{1}{(1 + 1/\rho)^8}, \quad \frac{|\nabla^2 \xi(\mathbf{r})|}{k^2(\mathbf{r})} = \frac{2}{(k_0 r_s)^2} \frac{1}{\rho^4} \frac{(1 + 1/\rho^2)}{(1 + 1/\rho)^8}.\tag{56}$$

Note the above ratios are finite for all values  $1 \leq \rho < \infty$  (the valid range of the isotropic scaled radius  $\rho$ ), even though the numerators and denominators of the left hand sides of Eq. (56) each separately diverge as  $\rho \rightarrow 1$ . From Eq. (55),  $k(\rho) \xrightarrow{\rho \rightarrow \infty} k_0$ , so that  $k_0$  is the usual wavenumber of the particle at spatial infinity. For ordinary wavelengths,  $k_0 r_s \gg 1$ , since  $r_s \sim \text{km}$ , so the above ratios remain incredibly small even down to  $\rho = 1$ . Even if we were to consider ultra long wavelengths such that  $k_0 r_s \sim 1$ , the ratios in Eq. (56) could still be made small for values of  $\rho \sim 2$ , i.e.  $r = 2r_s$ , the Schwarzschild radius. In this case, the term  $\eta(\mathbf{r})$  arising from the covariant derivative term  $g^{\mu\nu}\Gamma_{\mu\nu}^\lambda$  is essentially negligible compared to  $k^2(\mathbf{r})$  all the down to the Schwarzschild radius, and for all intents and purposes, our wave equation for massless particles is of the form

$$\nabla^2 u(\mathbf{r}) + k^2(\mathbf{r}) u(\mathbf{r}) = 0.\tag{57}$$

However, in the next section it causes no great difficulty to carry terms involving  $\eta(\mathbf{r})$  along formally.

## 5.5 Quantum corrections

The next order  $\hbar$  corrections to the phase and amplitude of the scalar wave-function arise from Eq. (37c). Using Eq. (46) we can write a first order

equation for  $dS_2/dl$  whose solution is given by

$$S_2(l, \alpha, \beta) = - \int^l \frac{dl}{2p} \left[ \frac{1}{2} \nabla^2 \mu(\mathbf{r}) - \frac{1}{4} (\nabla \mu(\mathbf{r}))^2 + \eta(\mathbf{r}) \right] \equiv -\mathcal{S}_2(l, \alpha, \beta). \quad (58)$$

Since  $S_2$  is purely real, this is an  $O(\hbar^2)$  correction to the phase.

The remaining equations Eq. (37d) can be formally solved to give

$$S_n(l, \alpha, \beta) = \int^l \frac{dl}{2p} \left[ i \nabla^2 S_{n-1} - \sum_{j=1}^{n-1} \nabla S_j \cdot \nabla S_{n-j} \right] \quad (59)$$

where the terms in the [ ], involve only the previously determined quantities  $\{S_1, S_2, \dots, S_{n-1}\}$ . The first correction to the amplitude is  $O(\hbar^2)$  and is given by  $S_3$  via

$$S_3(l, \alpha, \beta) = i \int^l \frac{dl}{2p} [\nabla^2 S_2 - \nabla \mu(\mathbf{r}) \cdot \nabla S_2] \equiv i\mathcal{S}_3(l, \alpha, \beta). \quad (60)$$

Putting this all together, we have to  $O(\hbar^2)$  in the phase and amplitude

$$\begin{aligned} \psi_3(\mathbf{r}, t) = \sqrt{\frac{\Phi(\mathbf{r})}{\Omega(\mathbf{r})}} \phi_3(\mathbf{r}, t) &= A \sqrt{\frac{\Phi(\mathbf{r})}{\Omega(\mathbf{r})}} \sqrt{\frac{p(\mathbf{r}_0)J(\mathbf{r}_0)}{p(\mathbf{r})J(\mathbf{r})}} \exp[-\hbar^2 \mathcal{S}_3(\mathbf{r})] \\ &\exp \left[ \frac{i}{\hbar} \left( \int^{\mathbf{r}} \mathbf{p}(\mathbf{x}) \cdot d\mathbf{x} - E t - \hbar^2 \mathcal{S}_2(\mathbf{r}) \right) \right], \end{aligned} \quad (61)$$

where the subscript 3 on  $\psi_3(\mathbf{r}, t)$  indicates that we have carried out the WKB expansion to  $S_{n=3}$ .

Eq. (61) reveals that for spin 0 particles the quantum phase

$$S/\hbar = \frac{1}{\hbar} \left( \int^{\mathbf{r}} \mathbf{p}(\mathbf{x}) \cdot d\mathbf{x} - E t \right), \quad (62)$$

proposed by Stodolsky is only the lowest order (in  $\hbar$ ) approximation to the full phase. This is in stark contrast to the phase of the Dirac wavefunction, for which the phase Eq. (62) is exact. The failure of Eq. (62) to be the exact phase for spin 0 particles is directly attributable to the presence of second order spatial derivatives in the GRKGE. Since by Eq. (21), each component of the massless spin 1 field satisfies the GRKGE these remarks also hold for the electromagnetic field  $A_\mu(x)$ . For the spin values considered in this work, only spin 1/2 particles satisfying the GRDE, which contain first order spatial derivatives, have the phase of wavefunction given *exactly* by Eq. (62).

## 6 The Optical-Mechanical Analogy

In this section we elucidate the optical-mechanical analogy for which the path of a particle in a gravitational field can be considered to arise from a spatially varying effective index of refraction [9, 10, 11]. To lowest order in  $\hbar$  for spin 0,1 particles, and exactly for the case of spin 1/2 particles, this is the classical path the quantum particle follows. For static metrics in isotropic coordinates the magnitude of the momentum  $p$  of the particle is function of position via Eq. (39),  $p(\mathbf{r}) = (E/c_0) N(\mathbf{r})$ . Using this and the quantization conditions Eq. (9) in the mass shell condition Eq. (8) we can define the phase velocity  $v_{phase}$  and group velocity  $v_g$  as

$$v_{phase} = \frac{\omega}{k} = \frac{c_0}{N(\mathbf{r})}, \quad (63a)$$

$$v_g = \frac{\partial \omega}{\partial k} = v(\mathbf{r}) \quad (63b)$$

where the velocity  $v(\mathbf{r})$  is given by [9]

$$v(\mathbf{r}) = \frac{c_0}{n(\mathbf{r})} \quad \text{massless particle,} \quad (64a)$$

$$v(\mathbf{r}) = \frac{c_0}{n(\mathbf{r})} \sqrt{1 - \left( \frac{\Omega(\mathbf{r}) m c_0^2}{E} \right)^2} \quad \text{massive particle.} \quad (64b)$$

Eq. (63a) allows us to identify  $N(\mathbf{r})$  defined in Eq. (41) as the index of refraction for massive de Broglie waves. Similarly, Eq. (64a) and the limit  $N(\mathbf{r}) \xrightarrow{m \rightarrow 0} n(\mathbf{r})$  allows us to identify  $n(\mathbf{r})$  as the index of refraction for massless de Broglie waves.

We can also add weight to the assertion that  $N$  is an index of refraction by deriving a geometrical optics ray equation for  $N$ . We begin with our ray equation Eq. (43), multiply through by  $p$  and differentiate both sides with respect to the arclength  $l$  to obtain

$$\frac{d}{dl} \left( p \frac{d\mathbf{x}}{dl} \right) = \frac{d}{dl} (\nabla S_0) = \nabla \frac{dS_0}{dl} = \nabla p,$$

where we have used Eq. (44) in the last equality [33]. Substituting in  $(E/c_0) N$  for  $p$  on both sides of the above equation yields the geometrical optics ray equation for the index of refraction  $N$ ,

$$\frac{d}{dl} \left( N \frac{d\mathbf{x}}{dl} \right) = \nabla N. \quad (65)$$

Eq. (65) is also directly derivable from the variational principle [34].

$$\delta \int N dl = 0. \quad (66)$$

Further discussion on the interpretation of  $N(\mathbf{r})$  as the index of refraction for massive de Broglie waves can be found in [11].

## 7 The Quantum Phase for Neutrino Oscillations

As an application, we will calculate the neutrino oscillation formula based on the QMP expressions calculated above, for the assumed mixing of massive neutrinos following Fornengo *et al.* [2]. Our interests are two-fold: (1) an example illustrating the explicit computation of the QMP and (2) an interpretation of the QMP in terms of an effective index of refraction.

In flat spacetime neutrinos (spin 1/2) produced by the weak interaction process are created in a flavor eigenstate  $|\nu_\alpha\rangle$  which is a superposition of mass eigenstates  $|\nu_k\rangle$  i.e.  $|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$ . Here  $U$  is the unitary matrix which mixes the different neutrino mass fields. What actually propagates is the mass eigenstates, whose energy  $E_k$  and momentum  $\mathbf{p}_k$  are related by the mass shell condition  $E_k^2 = (\mathbf{p}_k c_0)^2 + (m_k c_0^2)^2$ , and are determined at the production spacetime point  $A$ . In general,  $E_k$ ,  $\mathbf{p}_k$  and  $m_k$  are different for the different mass states. In flat spacetime each of the mass eigenstates propagates as  $|\nu_k(\mathbf{r}, t)\rangle = \exp(iS_k/\hbar)|\nu_k\rangle$  where  $S_k = \mathbf{p}_k \cdot \mathbf{x} - E_k t$ .

Neutrino oscillations occur because the different mass states propagate differently due to the differences in their energies and momenta. When they arrive at a detector located at a spacetime point  $B$  which detects flavor eigenstates via the weak interaction process, they have developed a relative phase shift. Interference between the different mass eigenstates at  $B$  produces the neutrino oscillations. One assumes the mass eigenstates are produced by some coherent process at the spacetime point  $A$  and that they are detected at the same spacetime point  $B$ . The probability that the neutrino  $|\nu_e\rangle$  produced at  $A$  is detected as  $|\nu_\mu\rangle$  at  $B$  is given by (for two generations)  $\mathcal{P}(\nu_e \rightarrow \nu_\mu) = |\langle \nu_e | \nu_\mu(B) \rangle|^2 = \sin^2 \theta \sin^2(S_{12}/2\hbar)$  where  $\theta$  is a mixing angle,  $S_k$  are the phases acquired by the mass eigenstates, and  $S_{12} = S_1 - S_2$ .

Fornengo *et al.* use the Stodolsky expression for the QMP as given by Eq. (1), reasoning that this form of the phase is valid independent of the particle's spin. From our result Eq. (27) for spin 1/2 particles, we see that this is indeed the correct choice, though not from their original premise. Since the neutrinos all begin at the spacetime point  $A$  and are detected at

the spacetime point  $B$ , they all experience the same amplitude change as given in Eq. (27), so we can ignore it. We write the phase (without  $\hbar$ ) as

$$S = \int_A^B \left( E \frac{dt}{dr} - p_r \right) dr \quad (67)$$

Fornengo *et al.* use the following procedure. They compute the phase  $S$  for a radial light-like trajectory with the mass shell condition given by Eq. (2). This assumes that one is considering ultra-relativistic neutrinos with  $E_k \gg m_k c_0^2$ . For radial null geodesics we obtain the condition Eq. (7a) which gives  $dt/dr = n(\mathbf{r})/c_0$ . The radial momentum  $p_r$  is given by Eq. (39),  $p_r = (E_k/c_0) N(\mathbf{r}) = (E_k/c_0) [1 - (\Omega(\mathbf{r})m_k c_0^2/E_k)^2]^{1/2}$ . Noting that  $E_k/c_0$  is a constant momentum along the geodesic we have

$$S_k = \left( \frac{E_k}{c_0} \right) \int_{\mathbf{r}_A, t_A}^{\mathbf{r}_B, t_B} \left( n(\mathbf{r}) - N(\mathbf{r}) \right) dr \quad (68a)$$

$$= \left( \frac{E_k}{c_0} \right) \int_{\mathbf{r}_A, t_A}^{\mathbf{r}_B, t_B} n(\mathbf{r}) \left( 1 - \sqrt{1 - \left( \frac{\Omega(\mathbf{r})m_k c_0^2}{E_k} \right)^2} \right) dr. \quad (68b)$$

Under the condition of ultra-relativistic neutrinos, the second term under the radical is assumed small and the square root can be expanded to first order. For the case of the Schwarzschild metric we define  $\rho = r/r_s$  and carrying out the integral yields

$$S_k^{Schw} = \left( \frac{E_k}{c_0} \right) \frac{r_s}{2} \left( \frac{m_k c_0^2}{E_k} \right)^2 \left[ |\rho_B - \rho_A| + \left| \frac{1}{\rho_B} - \frac{1}{\rho_A} \right| + \dots \right]. \quad (69)$$

This is essentially the form that Fornengo *et al.* write down in their paper ([2], Eq. (39)) except that here we use isotropic coordinates, and they make the further ultra-relativistic approximation  $E_k \approx E_0 + \mathcal{O}(m_k^2 c_0^4/2E_0)$  where  $E_0$  is the energy at spatial infinity for a massless particle. Oscillations then occur at phase shifts proportional to  $(\Delta m_{kj}/2E_0) |\rho_B - \rho_A| + \mathcal{O}(\rho^{-1})$  where  $\Delta m_{kj} = m_k^2 - m_j^2$ .

We note two points. First, the gravitational effects are implicit in Eq. (69) since  $\rho$  is the scaled coordinate distance. In the presence of gravity, the neutrino propagates over the proper distance  $L_p$  given by  $L_p = \int \sqrt{g_{rr}} dr = r_s \int (1 + 1/\rho)^2 d\rho = r_s [\rho - 1/\rho + 2 \ln \rho]$ . Second, Eq. (68a) allows us to interpret the phase  $S_k$  as calculated in the procedure of Fornengo *et al.* as the integrated “optical path difference” resulting from the difference between the index of refraction for a massless particle  $n(\mathbf{r})$  and a massive particle  $N(\mathbf{r})$

of momentum  $E_k/c_0$ . Note that if the neutrino was massless,  $N(\mathbf{r}) \rightarrow n(\mathbf{r})$  and  $S_k = 0$ . The authors chose their method of calculation over that of calculating the phase along the classical trajectory as in Ref. [35] even though the final results agree. In the later case, the classical trajectories of different massive neutrinos reaching the detection point at the same time must start at the production point at different times. Thus, there are initial phases for the wave functions that must be added in “by hand.” Fornengo’s *et al.* approach calculates the interference between mass eigenstates produced at the same spacetime point  $A$  and detected at the same spacetime point  $B$  connected by a null geodesics. We see that their “mixed” approach can be interpreted as the accumulation of phase due to the difference between massless and massive de Broglie waves.

If the calculations above were repeated for either a massless spin 0 particle or say a photon, the derivation would proceed the same except the mass would be set to zero in Eq. (68b). This would imply the lowest order contribution to the phase, i.e. the “classical phase” Eq. (67), would be identically zero. The next contribution to the phase would come from the quantum correction terms  $\exp(-i\hbar\mathcal{S}_2(\mathbf{r}))$  in Eq. (61) where  $\mathcal{S}_2(\mathbf{r})$  is defined by Eq. (58).

## 8 Conclusion

In this paper we have examined the proposal that the phase of the wave function for quantum mechanical particle in curved spacetime takes the form of Eq. (1), as put forth by Stodolsky [5]. We investigated the wave equations for spin 0, 1/2, 1 particles in the background of an arbitrary static gravitational field which can be written in isotropic coordinates and developed explicit plane wave-like solutions. We found that only for the case of spin 1/2 does the phase take the form of Eq. (1) *exactly*. This was directly attributable to the first order spatial derivative structure of the Dirac wave equation. For spin 0 and spin 1 particles the phase takes the form of Eq. (1) only to lowest order in  $\hbar$ , due to the second order spatial derivative structure of the corresponding wave equations. We developed a WKB solution for spin 0 particles which is also applicable for spin 1 particles. We noted that in a gravitational field, the wave function for the generally covariant extension of the Dirac equation is not necessarily in addition a solution to the curved spacetime Klein-Gordon equation, as is the case in flat spacetime. We find it very intriguing that in the presence of a gravitational field, the Dirac equation continues to admit exactly a generally covariant extension of a plane



wave-like solution with the phase given by Eq. (1), while the Klein-Gordon and massless spin 1 wave equations only do so to lowest order in  $\hbar$ .

For the case of spin 1/2 particles we calculated the quantum mechanical phase appropriate for neutrino flavor oscillations for radial geodesics. For spin 0 and spin 1 particles the phase of the quantum wave function is predominantly the classical phase as given by Eq. (1), with higher order quantum corrections. For most applications, especially those involving the solar system, the form of the quantum mechanical wave function in curved spacetime assumed by Stodolsky would be essentially correct (except that the amplitude would vary spatially) for all practical calculations. However, for the case of massless spin 0 and spin 1 particles and radial geodesics, we showed that the classical phase is zero and the higher order  $\hbar$  WKB phases would be the dominant contribution.

For all the wave equations discussed in this paper we drew an analogy for the geodesic path followed by a quantum particle in a static gravitational field to motion in a medium with a spatially varying effective index of refraction. By examining the momentum of the quantum particle, we were able to define an effective index of refraction  $n(\mathbf{r})$  and  $N(\mathbf{r})$  for massless for massive de Broglie waves, respectively. This allows us to extend the classical optical-mechanical analogy to the quantum regime for arbitrary static, background gravitational fields which can be written in isotropic coordinates.

## A Dirac Gamma Matrices in Flat Spacetime

In this section we adhere to the notation of the main body of the text and use Latin indices  $\{a, b, c\}$  to indicate flat spacetime indices in the range  $\{0, 1, 2, 3\}$ . Latin indices in the middle of the alphabet  $\{i, j, k\}$  refer to spatial indices  $\{1, 2, 3\}$ . The Lorentz metric is given by  $\eta_{ab} = \text{diagonal}\{1, -1, -1, -1\}$ .

The defining relation for the  $4 \times 4$  Dirac gamma matrices is

$$\{\gamma^a, \gamma^b\} = \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \quad (\text{A.1})$$

$\gamma^0$  is unitary and Hermetian  $(\gamma^0)^2 = \mathbf{I}_{4 \times 4} = \gamma^0 \gamma^{0\dagger}$ , while the spatial gamma matrices  $\gamma^i$  are unitary  $(\gamma^i)^{-1} = (\gamma^i)^\dagger$  and anti-Hermetian  $(\gamma^i)^\dagger = -\gamma^i$ , with  $(\gamma^i)^2 = -\mathbf{I}_{4 \times 4} = \gamma^i \gamma^{i\dagger}$ . In this paper we use the explicit representation [31, 36]

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \quad \gamma = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \quad (\text{A.2})$$

where  $\mathbf{0}$  and  $\mathbf{I}$  are the  $2 \times 2$  zero and identity matrix respectively,  $\boldsymbol{\gamma} = \{\gamma^1, \gamma^2, \gamma^3\}$ , and  $\boldsymbol{\sigma} = \{\sigma^1, \sigma^2, \sigma^3\}$  are the  $2 \times 2$  Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.3})$$

The Pauli matrices have the property  $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$ . Here  $\epsilon^{ijk}$  is the Levi-Civita symbol with  $\epsilon^{123} = 1$  and anti-symmetric in all its indices. Another useful relationship is  $(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i \boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b})$  for any arbitrary pair of spatial vectors  $\mathbf{a}$  and  $\mathbf{b}$ .

In the derivation of the wave equation for arbitrary spin in Section 3 we introduced the anti-symmetric spin matrices

$$\sigma^{ab} \equiv \frac{i}{2} [\gamma^a, \gamma^b] = \frac{i}{2} (\gamma^a \gamma^b - \gamma^b \gamma^a). \quad (\text{A.4})$$

In the representation of Eq. (A.2) we have

$$\sigma^{0k} = i \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \sigma^{ij} = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \quad (\text{A.5})$$

## B Proof of Eq. (48)

In this appendix we will prove Eq. (48)

$$\frac{\nabla^2 S_0}{p} = \frac{d \ln(pJ)}{dl} \quad (\text{B.1})$$

where  $J$  is the Jacobian of the transformation from curvilinear coordinates  $(l, \alpha, \beta)$  (which describe the wave fronts  $S_0(\mathbf{r})$ ) to Cartesian coordinates. In order to prove this relation, we must first prove the following lemma, Eq. (50).

### B.1 Lemma: Proof of Eq. (50)

We want to prove

$$\partial_l J = J \nabla \cdot \left( \frac{\nabla S_0}{p} \right) = J \nabla \cdot \frac{d\mathbf{x}}{dl} \quad (\text{B.2})$$

which states that the logarithmic derivative of the Jacobian  $J$  along a congruence of ray trajectories is equal to the divergence of the tangent vector field of the congruence.

A standard result proved in most relativity books (see for e.g. [12] p242-243, [37] p93-94), is that for any matrix  $a_{ij}$ , with determinant  $a$  and inverse,  $a^{ij} = A^{ji}/a$ , where  $A^{ji}$  is the signed cofactor of  $a_{ij}$ , we have

$$\frac{\partial a}{\partial x^k} = a a^{ji} \frac{\partial a_{ij}}{\partial x^k} = a a^{ij} \frac{\partial a_{ij}}{\partial x^k} \text{ for } a_{ij} \text{ symmetric.} \quad (\text{B.3})$$

Let

$$J = \left| \frac{\partial \mathbf{x}}{\partial (l, \alpha, \beta)} \right|, \quad (\text{B.4})$$

be the determinant of the transformation matrix from Cartesian coordinates  $\mathbf{x}$ , to the curvilinear ray coordinates  $(l, \alpha, \beta)$ . Writing out Eq. (B.3) with  $a_{ij} \rightarrow J_{ij} = \partial x^i / \partial x'^j$ , with  $x^i$  as Cartesian coordinates and  $x'^i$  as curvilinear coordinates yields

$$\frac{\partial J}{\partial x'^k} = J \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^k} \left( \frac{\partial x^i}{\partial x'^j} \right) = J \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j} \left( \frac{\partial x^i}{\partial x'^k} \right) = J \frac{\partial}{\partial x^i} \left( \frac{\partial x^i}{\partial x'^k} \right), \quad (\text{B.5})$$

where in the second equality we have interchanged the order of the differentiations  $\partial/\partial x'^k$  and  $\partial/\partial x'^j$ , and have used the chain rule in the last equality. If we set  $k = 1$  with  $x'^1 = l$ , the term in the last ( ) above is just  $d\mathbf{x}/dl$ , in component form. Thus, with  $k = 1$ , Eq. (B.5) is just Eq. (B.2) in component form.

## B.2 Proof of Eq. (48):

Using Eq. (B.2), we want to show

$$\nabla^2 S_0 / p = d \ln (pJ) / dl. \quad (\text{B.6})$$

Using the ray equation Eq. (43) in the form

$$\frac{d\bar{\mathbf{x}}}{dl} = \frac{\nabla S_0(\mathbf{r})}{p} \quad (\text{B.7})$$

we can write the first equality in Eq. (B.2) as (expanding out the divergence)

$$\frac{d \ln J}{dl} = -\frac{\nabla S_0}{p} \cdot \frac{\nabla p}{p} + \frac{\nabla^2 S_0}{p} = -\frac{d\bar{\mathbf{x}}}{dl} \cdot \nabla \ln p + \frac{\nabla^2 S_0}{p} = -\frac{d \ln p}{dl} + \frac{\nabla^2 S_0}{p}. \quad (\text{B.8})$$

Solving for  $\nabla^2 S_0 / p$  in Eq. (B.8) yields the desired result Eq. (B.6).

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$$\left[1/2m_0 (\hat{\mathbf{p}} - e/c_0\mathbf{A})^2 + e\phi + \mu_B \hat{\boldsymbol{\sigma}} \cdot \mathbf{B}\right] \psi = 0.$$
 See for example W. Greiner, *Quantum Mechanics, An Introduction*, 2nd ed. p308-311, 323-333 (Springer Verlag, N.Y., 1994). If we had allowed for the inclusion of an external electromagnetic field  $F_{\mu\nu}$  through the minimal coupling scheme  $\Omega_\mu \rightarrow \Omega_\mu + i\xi A_\mu$ , with  $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$  Eq. (19) would contain an extra term of the form  $-1/2\xi F_{\mu\nu}\sigma^{\mu\nu}\psi$ .
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